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Thermal postbuckling analysis of imperfect Reissner-Mindlin plates on softening nonlinear elastic foundations

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Abstract. A thermal postbuckling analysis is presented for a simply supported, moderately thick rectangular plate subjected to uniform or nonuniform tent-like temperature loading and resting on a softening nonlinear elastic foundation. The initial geometrical imperfection of the plate is taken into account. The formulations are based on the Reissner-Mindlin plate theory considering the first-order shear-deformation effect, and including plate-foundation interaction and thermal effects. The analysis uses a deflection-type perturbation technique to determine the thermal buckling loads and postbuckling equilibrium paths. Numerical examples are presented that relate to the performances of perfect and imperfect, moderately thick plates resting on softening nonlinear elastic foundations. The effects played by foundation stiffness, transverse shear deformation, plate aspect ratio, thermal load ratio and initial geometrical imperfections are studied. Typical results are presented in dimensionless graphical form and exhibit interesting imperfection sensitivity.

Key words: elastic foundation, moderately thick plate, perturbation method, thermal postbuckling, structural stability

1. Introduction

In the thermal analysis of concrete pavements of roads and airfields, the problem is usually simplified and analyzed as moderately thick rectangular plates supported by an elastic foundation. These plates may have significant and unavoidable initial geometrical imperfections. Due to boundary constraints, varying temperature environments typically induce stress, with ensuing buckling. Therefore, there is a need to understand the thermal buckling and postbuckling behavior of imperfect shear-deformable rectangular plates resting on elastic foundations.

Although a limited number of publications have appeared in the literature on the thermal buckling of thick plates subjected to temperature distribution, investigation of the thermal postbuckling response of thick plates is very limited. Thermal buckling loads for initially stressed transversely isotropic and antisymmetrically cross-ply laminated thick plates were evaluated by means of the Galerkin method by Chen *et al.* [1] and by Yang and Shieh [2]. Thermal buckling analyses of composite laminated thick plates subjected to uniform or nonuniform temperature loading have been made by Tauchert [3], Sun and Hsu [4] and Chen *et al.* [5].

Noor and Peters [6, 7] and Noor *et al.* [8] calculated buckling loads and postbuckling load-deflection curves of perfect, symmetrically laminated plates subjected to combined axial load and a uniform temperature distribution. Librescu and Souza [9] analyzed postbuckling of an imperfect, shear-deformable, transversely isotropic plate under combined thermal and compressive edge loading. Shen and Zhu [10] analyzed the thermal postbuckling of perfect and imperfect, moderately thick plates subjected to uniform or nonuniform tent-like temperature distribution using the deflection-type perturbation technique.

For elastic foundations, Raju and Rao [11] calculated the thermal postbuckling response of a thin isotropic square plate resting on a Winkler elastic foundation by the finite-element method. Dumir [12] analyzed the thermal postbuckling of a thin isotropic rectangular plate resting on a Pasternak-type elastic foundation using the Galerkin method, but his numerical results were only for the Winkler elastic foundation case.

The softening effect of some types of nonlinear foundations has been known for a long time and the initial postbuckling problem of imperfect thin plates resting on such foundations has been studied by Reissner [13]. Recently, Shen [14] gave the theoretical investigation of the postbuckling response of in-plane compressed, perfect and imperfect, isotropic and orthotropic thin plates resting on softening nonlinear elastic foundations. This work was extended to composite laminated plates by Shen and Williams [15] using the classical laminated plate theory. Also recently, Librescu and Lin [16] analyzed the postbuckling of imperfect, shear-deformable transversely isotropic flat and curved panels subjected to complex mechanical loading and resting on nonlinear elastic foundations. To the best of the author's knowledge, no papers deal with the thermal postbuckling of imperfect Reissner-Mindlin plates subjected to nonuniform temperature loading and resting on softening nonlinear elastic foundations.

Therefore, the present work focuses attention on the thermal postbuckling of moderately thick plates subjected to uniform or nonuniform tent-like temperature loading and resting on softening nonlinear elastic foundations. The formulations are based on the Reissner-Mindlin first-order shear deformation plate theory, including plate-foundation interaction and thermal effects. The analysis uses a deflection-type perturbation technique to determine the required thermal buckling loads and postbuckling equilibrium paths. The material properties are assumed to be independent of temperature. The initial geometrical imperfection of the plate is taken into account but, for simplicity, its form is taken as the initial buckling mode of the plate.

2. Analytical formulation

Consider a moderately thick rectangular plate of length a, width b and thickness t simply supported on its four edges. The plate is subjected to thermal loading and rests on an elastic foundation. The foundation is assumed to be an attached foundation, that means no part of the plate lifts off the foundation in the postbuckled regime. The load-displacement relationship of the foundation is assumed to be $p = \overline{K}_1\overline{W} - \overline{K}_3\overline{W}^3$, as used for imperfect columns by Amazigo *et al.* [17], where p is the force per unit area, $\overline{K}_1$ is the Winkler foundation stiffness and $\overline{K}_3$ is the softening nonlinear elastic foundation stiffness. $\overline{U}, \overline{V}$ and $\overline{W}$ are the plate displacements parallel to a right-hand set of axes (X, Y, Z), where X is longitudinal and Z is perpendicular to the plate. Denoting the initial deflection by $\overline{W}^*(X, Y)$, let $\overline{W}(X, Y)$ be the additional deflection, and $\overline{F}(X, Y)$ be the stress function for the stress resultants, so that $N_x = \overline{F}_{,yy}$, $N_y = \overline{F}_{,xx}$ and $N_{xy} = -\overline{F}_{,xy}$.

From the Reissner-Mindlin plate theory considering the first-order shear-deformation effect, including plate-foundation interaction and thermal effects, the governing differential equations of such plates are

$$D\nabla^4\overline{W} + \nabla^2 M^T = \left(1 - \frac{D}{\kappa^2 G\, t}\nabla^2\right)[L(\overline{W} + \overline{W}^*, \overline{F}) - (\overline{K}_1\overline{W} - \overline{K}_3\overline{W}^3)], \tag{1}$$

$$\nabla^4\overline{F} + (1 - \nu)\nabla^2 N^T = -\tfrac{1}{2}\, E\, t\, L(\overline{W} + 2\overline{W}^*, \overline{W}), \tag{2}$$

where

$$\nabla^4 = \frac{\partial^4}{\partial X^4} + 2\frac{\partial^4}{\partial X^2 \partial Y^2} + \frac{\partial^4}{\partial Y^4}, \quad \nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2},$$

$$L(\quad) = \frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial Y^2} - 2\frac{\partial^2}{\partial X \partial Y}\frac{\partial^2}{\partial X \partial Y} + \frac{\partial^2}{\partial Y^2}\frac{\partial^2}{\partial X^2},$$

in which D is flexural rigidity and $D = Et^3/12(1-\nu^2)$. E is Young's modulus, G is the in-plane shear modulus and ν is the Poisson's ratio. Also κ^2 is the shear factor, which accounts for the nonuniform part of the shear-strain distribution through the plate thickness, and for Reissner plate theory $\kappa^2 = 5/6$, while for Mindlin plate theory $\kappa^2 = \pi^2/12$.

The nonuniform tent-like temperature rise is

$$T(X,Y,Z) = \begin{cases} T_0 + 2T_1 Y/b & 0 \leqslant Y \leqslant b/2 \\ T_0 + 2T_1(1 - Y/b) & b/2 \leqslant Y \leqslant b, \end{cases} \tag{3}$$

where T_0 is the uniform temperature rise, and T_1 is the temperature gradient.

The thermal force and moment are defined by

$$(N^T, M^T) = \frac{E\alpha}{1-\nu}\int_{-t/2}^{+t/2}(1,Z)T(X,Y,Z)\,\mathrm{d}Z \tag{4}$$

in which α is thermal expansion coefficient for a plate. Because of Equations (3) and (4), it is noted that the thermal moment $M^{\mathrm{T}} = 0$ and $\nabla^2 N^{\mathrm{T}} = 0$.

The average end-shortening relationship is

$$\frac{\Delta_x}{a} = -\frac{1}{abt}\int_{-t/2}^{+t/2}\int_0^b\int_0^a \frac{\partial \overline{U}}{\partial X}\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = -\frac{1}{ab}\int_0^b\int_0^a\Bigg[\frac{1}{Et}\left(\frac{\partial^2 \overline{F}}{\partial Y^2} - \nu\frac{\partial^2 \overline{F}}{\partial X^2}\right)$$
$$-\frac{1}{2}\left(\frac{\partial \overline{W}}{\partial X}\right)^2 - \frac{\partial \overline{W}}{\partial X}\frac{\partial \overline{W}^*}{\partial X} + \frac{1}{Et}(1-\nu)N^{\mathrm{T}}\Bigg]\,\mathrm{d}X\,\mathrm{d}Y, \tag{5a}$$

$$\frac{\Delta_y}{b} = -\frac{1}{abt}\int_{-t/2}^{+t/2}\int_0^a\int_0^b \frac{\partial \overline{V}}{\partial Y}\,\mathrm{d}Y\,\mathrm{d}X\,\mathrm{d}Z = -\frac{1}{ab}\int_0^a\int_0^b\Bigg[\frac{1}{Et}\left(\frac{\partial^2 \overline{F}}{\partial X^2} - \nu\frac{\partial^2 \overline{F}}{\partial Y^2}\right)$$
$$-\frac{1}{2}\left(\frac{\partial \overline{W}}{\partial Y}\right)^2 - \frac{\partial \overline{W}}{\partial Y}\frac{\partial \overline{W}^*}{\partial Y} + \frac{1}{Et}(1-\nu)N^{T}\Bigg]\,\mathrm{d}Y\,\mathrm{d}X. \tag{5b}$$

All the edges are assumed to be simply supported and to be restrained against expansion in the X- and Y-directions, respectively. The tangential motion parallel to the immovable edges is unrestrained, *i.e.* the membrane shear forces are zero-valued (see [12]). Hence the boundary conditions are

$$X = 0, a: \qquad \overline{W} = \overline{U} = 0, \qquad N_{xy} = \overline{M}_x = 0, \tag{6a,b,c,d}$$

$$Y = 0, b: \qquad \overline{W} = \overline{V} = 0, \qquad N_{xy} = \overline{M}_y = 0, \tag{6e,f,g,h}$$

where $\overline{M}_x$ and $\overline{M}_y$ are, respectively, the bending moments per unit width and per unit length of the plate.

Equations (1)–(6) are the governing equations describing the required large deflection thermal postbuckling response of the plate.

3. Analytical method and asymptotic solutions

Let $\lambda_{\mathrm{T}} = 12(1+\nu)b^2\alpha T_i/\pi^2 t^2$ $(i = 0$ for uniform temperature distribution, and $i = 1$ otherwise) and introducing the dimensionless quantities (in which the alternative forms k_1 and k_3 are not needed until the numerical examples are considered)

$$x = \pi X/a, \quad y = \pi Y/b, \quad \beta = a/b, \quad (W, W^*) = (\overline{W}, \overline{W}^*)[12(1-\nu^2)]^{1/2}/t,$$

$$F = \overline{F}/D, \quad \gamma = \pi^2 D/\kappa^2 a^2\, G\, t, \quad (M_x, M_y) = (\overline{M}_x, \overline{M}_y)a^2[12(1-\nu^2)]^{1/2}/\pi^2\, D\, t,$$

$$(K_1, k_1) = (a^4, b^4)\overline{K}_1/\pi^4 D, \quad (K_3, k_3) = (a^4, b^4)\overline{K}_3/\pi^4\, E\, t,$$

$$(\delta_x, \delta_y) = (\Delta_x/a, \Delta_y/b)12(1-\nu^2)b^2/4\pi^2 t^2 \tag{7a-j}$$

enables the nonlinear Equations (1) and (2) to be written in dimensionless form as

$$\overline{\nabla}^4 W + [1-\gamma\overline{\nabla}^2][K_1 W - K_3 W^3] = \beta^2[1-\gamma\overline{\nabla}^2]L(W+W^*, F), \tag{8}$$

$$\overline{\nabla}^4 F = -\tfrac{1}{2}\beta^2 L(W + 2W^*, W), \tag{9}$$

where

$$\overline{\nabla}^4 = \frac{\partial^4}{\partial x^4} + 2\beta^2\frac{\partial^4}{\partial x^2\partial y^2} + \beta^4\frac{\partial^4}{\partial y^4}, \quad \overline{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \beta^2\frac{\partial^2}{\partial y^2},$$

$$L(\ ) = \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} - 2\frac{\partial^2}{\partial x\partial y}\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial x^2},$$

The unit end-shortening relationships become

$$\delta_x = -\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\beta^2\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right) - \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 - \frac{\partial W}{\partial x}\frac{\partial W^*}{\partial x} + (1-\nu)\lambda_{\mathrm{T}}\beta^2 C_{11}\right]\mathrm{d}x\,\mathrm{d}y \tag{10a}$$

$$\delta_y = -\frac{1}{4\pi^2\beta^2}\int_0^\pi\int_0^\pi\left[\left(\frac{\partial^2 F}{\partial x^2} - \nu\beta^2\frac{\partial^2 F}{\partial y^2}\right) - \frac{1}{2}\beta^2\left(\frac{\partial W}{\partial y}\right)^2 - \beta^2\frac{\partial W}{\partial y}\frac{\partial W^*}{\partial y} + (1-\nu)\lambda_{\mathrm{T}}\beta^2 C_{11}\right]\mathrm{d}y\,\mathrm{d}x. \tag{10b}$$

Note that Equations (8) and (9) are identical to those of Reissner-Mindlin plates under pure axial compression and resting on softening nonlinear elastic foundations (see [18]), but Equation (10) contains terms in $\lambda_{\rm T}$. In Equation (10), for the uniform thermal loading case, $C_{11} = 1{\cdot}0$ and $\lambda_{\rm T} = 12(1+\nu)b^2\alpha T_0/\pi^2 t^2$, whereas for the tent-like temperature loading case, $C_{11} = (T_0/T_1 + 1/2)$ and $\lambda_{\rm T} = 12(1+\nu)b^2\alpha T_1/\pi^2 t^2$.

The condition $\overline{U} = 0$ (or $\overline{V} = 0$) for the immovable edges $X = 0, a$ (or $Y = 0, b$) may be expressed in an average sense as (see [16])

$$\int_0^b \int_0^a \frac{\partial \overline{U}}{\partial X}\,{\rm d}X\,{\rm d}Y = 0 \qquad \left(\text{or } \int_0^a \int_0^b \frac{\partial \overline{V}}{\partial Y}\,{\rm d}Y\,{\rm d}X = 0\right).$$

Then the boundary conditions of Equation (6) become

$$x = 0, \pi: \qquad W = 0, \quad \delta_x = 0, \quad F_{,xy} = 0, \quad M_x = 0, \tag{11a,b,c,d}$$

$$y = 0, \pi: \qquad W = 0, \quad \delta_y = 0, \quad F_{,xy} = 0, \quad M_y = 0. \tag{11e,f,g,h}$$

Applying Equations (8)–(11), the thermal postbuckling behavior of Reissner-Mindlin plates resting on a softening nonlinear elastic foundation can be determined by the perturbation technique suggested in [19]. The essence of this procedure, in the present case, is to assume that

$$W(x,y) = \sum_{j=1} \varepsilon^j w_j(x,y), \quad F(x,y) = \sum_{j=0} \varepsilon^j f_j(x,y), \tag{12a,b}$$

where ε is a small perturbation parameter, and the first term of $w_j(x,y)$ is assumed to have the form

$$w_1(x,y) = A_{11}^{(1)} \sin mx \ \ \sin ny \tag{13}$$

The initial geometrical imperfection is assumed to have a similar form to $w_1(x,y)$, *i.e.*

$$W^*(x,y) = \varepsilon A_{11}^* \sin mx \ \ \sin ny = \varepsilon\mu A_{11}^{(1)} \sin mx \ \ \sin ny, \tag{14}$$

where μ is the imperfection parameter.

Substituting Equation (12) in Equations (8) and (9), collecting the terms of the same order of ε, we derive a system of perturbation equations which can be written as

$$0(\varepsilon^0): \ \overline{\nabla}^4 f_0 = 0, \tag{15}$$

$$0(\varepsilon^1): \ \overline{\nabla}^4 w_1 + (1-\gamma\overline{\nabla}^2)[K_1 w_1 - \beta^2 L(w_1 + W^*, f_0)] = 0, \quad \overline{\nabla}^4 f_1 = 0, \tag{16,17}$$

$$0(\varepsilon^2): \ \overline{\nabla}^4 w_2 + (1-\gamma\overline{\nabla}^2)[K_1 w_2 - \beta^2 L(w_2, f_0)] = (1-\gamma\overline{\nabla}^2)\beta^2 L(w_1 + W^*, f_1), \tag{18}$$

$$\overline{\nabla}^4 f_2 = -\tfrac{1}{2}\beta^2 L(w_1 + 2W^*, w_1), \tag{19}$$

$$0(\varepsilon^3): \ \overline{\nabla}^4 w_3 + (1-\gamma\overline{\nabla}^2)[K_1 w_3 - \beta^2 L(w_3, f_0)]$$

$$= (1-\gamma\overline{\nabla}^2)[K_3 w_1^3 + \beta^2 L(w_2, f_1), + \beta^2 L(w_1 + W^*, f_2)], \tag{20}$$

$$\overline{\nabla}^4 f_3 = -\beta^2 L(w_1 + W^*, w_2), \tag{21}$$

By using Equations (13) and (14) to solve these perturbation equations of each order, we may determine the amplitudes of the terms $w_j(x,y)$ and $f_j(x,y)$ step by step and, as a result, the asymptotic solutions are obtained as

$$\begin{aligned} W &= \varepsilon[A_{11}^{(1)} \sin mx \;\; \sin ny] + \varepsilon^3[A_{13}^{(3)} \sin mx \;\; \sin 3ny + A_{31}^{(3)} \sin 3mx \;\; \sin ny \\ &\quad + A_{33}^{(3)} \sin 3mx \;\; \sin 3ny] + 0(\varepsilon^5), \end{aligned} \tag{22}$$

$$\begin{aligned} F &= -B_{00}^{(0)}\frac{y^2}{2} - b_{00}^{(0)}\frac{x^2}{2} + \varepsilon^2\left[-B_{00}^{(2)}\frac{y^2}{2} - b_{00}^{(2)}\frac{x^2}{2} + B_{20}^{(2)}\cos 2mx + B_{02}^{(2)}\cos 2ny\right] \\ &\quad +\varepsilon^4\Big[-B_{00}^{(4)}\frac{y^2}{2} - b_{00}^{(4)}\frac{x^2}{2} + B_{20}^{(4)}\cos 2mx + B_{02}^{(4)}\cos 2ny \\ &\qquad + B_{22}^{(4)}\cos 2mx \;\; \cos 2ny + B_{40}^{(4)}\cos 4mx + B_{04}^{(4)}\cos 4ny \\ &\qquad + B_{24}^{(4)}\cos 2mx \;\; \cos 4ny + B_{42}^{(4)}\cos 4mx \;\; \cos 2ny\Big] + 0(\varepsilon^5), \end{aligned} \tag{23}$$

where $B_{00}^{(i)}$ and $b_{00}^{(i)} (i = 0, 2, 4, \ldots)$ come from in-plane uniform compressive stresses induced by a temperature rise.

In Equations (22) and (23) all coefficients are related and can be written as functions of $A_{11}^{(1)}$. Illustrative ones are given here:

$$\begin{aligned} &\beta^2 B_{00}^{(0)} m^2 + b_{00}^{(0)} n^2\beta^2 = \frac{(m^2+n^2\beta^2)^2 + K_1[1+\gamma(m^2+n^2\beta^2)]}{(1+\mu)[1+\gamma(m^2+n^2\beta^2)]}, \\ &\beta^2 B_{00}^{(2)} m^2 + b_{00}^{(2)} n^2\beta^2 = \frac{1}{16(1+\mu)}[(m^4+n^4\beta^4)(1+\mu)(1+2\mu) - 9K_3](A_{11}^{(1)})^2, \\ &\beta^2 B_{00}^{(4)} m^2 + b_{00}^{(4)} n^2\beta^2 = -\frac{1}{256(1+\mu)}\left[\frac{b_{13}d_{13}}{g_{13}} + \frac{b_{31}d_{31}}{g_{31}} + 3K_3\frac{b_{33}}{g_{33}}\right](A_{11}^{(1)})^4, \\ &B_{20}^{(2)} = \frac{1}{32}\frac{n^2\beta^2}{m^2}(1+2\mu)(A_{11}^{(1)})^2, \quad B_{02}^{(2)} = \frac{1}{32}\frac{m^2}{n^2\beta^2}(1+2\mu)(A_{11}^{(1)})^2 \\ &A_{13}^{(3)} = \frac{1}{16}\frac{b_{13}}{g_{13}}(A_{11}^{(1)})^3, \quad A_{31}^{(3)} = \frac{1}{16}\frac{b_{31}}{g_{31}}(A_{11}^{(1)})^3, \quad A_{33}^{(3)} = \frac{1}{16}\frac{b_{33}}{g_{33}}(A_{11}^{(1)})^3, \\ &B_{20}^{(4)} = -\frac{1}{16}\frac{n^2\beta^2}{m^2}(1+\mu)A_{31}^{(3)}A_{11}^{(1)}, \quad B_{02}^{(4)} = -\frac{1}{16}\frac{m^2}{n^2\beta^2}(1+\mu)A_{13}^{(3)}A_{11}^{(1)}, \end{aligned} \tag{24a–j}$$

in which

$$b_{13} = [m^4(1+\mu)(1+2\mu) - 3K_3][1+\gamma(m^2+9n^2\beta^2)],$$

$$b_{31} = [n^4\beta^4(1+\mu)(1+2\mu) - 3K_3][1+\gamma(9m^2+n^2\beta^2)],$$

$$b_{33} = K_3[1+9\gamma(m^2+n^2\beta^2)], \qquad d_{13} = m^4[2(1+\mu)^2 + (1+2\mu)] - 9K_3,$$

$$d_{31} = n^4\beta^4[2(1+\mu)^2 + (1+2\mu)] - 9K_3,$$

$$g_{13} = \Theta_{13} - (\beta^2 B_{00}^{(0)} m^2 + b_{00}^{(0)} 9n^2\beta^2)[1+\gamma(m^2+9n^2\beta^2)],$$

$$g_{31} = \Theta_{31} - (\beta^2 B_{00}^{(0)} 9m^2 + b_{00}^{(0)} n^2\beta^2)[1+\gamma(9m^2+n^2\beta^2)],$$

$$g_{33} = \Theta_{33} - 9(\beta^2 B_{00}^{(0)} m^2 + b_{00}^{(0)} n^2\beta^2)[1+9\gamma(m^2+n^2\beta^2)],$$

$$\Theta_{13} = (m^2+9n^2\beta^2)^2 + K_1[1+\gamma(m^2+9n^2\beta^2)],$$

$$\Theta_{31} = (9m^2+n^2\beta^2)^2 + K_1[1+\gamma(9m^2+n^2\beta^2)],$$

$$\Theta_{33} = 81(m^2+n^2\beta^2)^2 + K_1[1+9\gamma(m^2+n^2\beta^2)]. \tag{25a–k}$$

Next, substituting Equations (22) and (23) in the boundary conditions $\delta_x = 0$ and $\delta_y = 0$, we have

$$\beta^2 B_{00}^{(0)} + \varepsilon^2\beta^2 B_{00}^{(2)} + \varepsilon^4\beta^2 B_{00}^{(4)} + \cdots = \beta^2\lambda_T C_{11} - \frac{1}{8}\frac{m^2+\nu n^2\beta^2}{1-\nu^2}(1+2\mu)(A_{11}^{(1)}\varepsilon)^2,$$

$$b_{00}^{(0)} + \varepsilon^2 b_{00}^{(2)} + \varepsilon^4 b_{00}^{(4)} + \cdots = \beta^2\lambda_T C_{11} - \frac{1}{8}\frac{\nu m^2+n^2\beta^2}{1-\nu^2}(1+2\mu)(A_{11}^{(1)}\varepsilon)^2. \tag{26a,b}$$

In Equations (26a, b), let ε approaches to zero, which leads to $\beta^2 B_{00}^{(0)} = b_{00}^{(0)}$.

Because of Equation (24a) we have

$$\beta^2 B_{00}^{(0)} = b_{00}^{(0)} = \frac{(m^2+n^2\beta^2)^2 + K_1[1+\gamma(m^2+n^2\beta^2)]}{(1+\mu)(m^2+n^2\beta^2)[1+\gamma(m^2+n^2\beta^2)]}. \tag{27}$$

If the maximum deflection of the plate is assumed to be at the point $(x, y) = (\pi/2m, \pi/2n)$, Equation (22) results in

$$W_m = \varepsilon A_{11}^{(1)} - \varepsilon^3(A_{13}^{(3)} + A_{31}^{(3)} - A_{33}^{(3)}) + \cdots, \tag{28}$$

where W_m is the dimensionless form of the maximum deflection of the plate.

The inverse form of Equation (28) can be written as

$$A_{11}^{(1)}\varepsilon = W_m + \frac{1}{16}\left(\frac{b_{13}}{g_{13}} + \frac{b_{31}}{g_{31}} - \frac{b_{33}}{g_{33}}\right)W_m^3 + \cdots. \tag{29}$$

Adding Equations (24a–c) and combining these with Equations (26a,b) and replacing the perturbation parameter $(A_{11}^{(1)}\varepsilon)$ with the maximum deflection W_m, then the thermal postbuckling equilibrium path can be written as

$$\lambda_T = \lambda_T^{(0)} + \lambda_T^{(2)} W_m^2 + \lambda_T^{(4)} W_m^4 + \cdots, \tag{30}$$

Table 1. Comparisons of various theories on the thermal buckling load $\lambda_T^* = \alpha T_0 \times 10^3$ for perfect isotropic square plates

b/t	Noor & Burton Ref. [20]	HSDPT[a]	Present $\kappa^2 = 5/6$	Present $\kappa^2 = \pi^2/12$	CPT[b]
100	0·1264	0·1265	0·1265	0·1265	0·1265
20	3·109	3·1194	3·1194	3·1188	3·1633
10	11·83	11·9782	11·9778	11·9694	12·6533
5	39·90	41·3175	41·2971	41·1969	50·6134

[a] Calculated using higher order shear deformation plate theory given in [21].
[b] Calculated using classical plate theory given in [22].

in which

$$(\lambda_T^{(0)}, \lambda_T^{(2)}, \lambda_T^{(4)}) = (S_0, S_2, S_4)/\beta^2(m^2 + n^2\beta^2)C_{11} \tag{31a}$$

and

$$S_0 = \Theta_1/(1+\mu)[1+\gamma(m^2+n^2\beta^2)], \quad S_2 = \Theta_2/16(1+\mu),$$
$$S_4 = (C_{24} - C_{44})/256(1+\mu), \quad \Theta_1 = (m^2+n^2\beta^2)^2 + K_1[1+\gamma(m^2+n^2\beta^2)]$$
$$\Theta_2 = [(3-\nu^2)(m^4+n^4\beta^4) + 4\nu m^2 n^2 \beta^2](1+\mu)(1+2\mu)/(1-\nu^2) - 9K_3,$$
$$C_{24} = 2\Theta_2\left(\frac{b_{13}}{g_{13}} + \frac{b_{31}}{g_{31}} - \frac{b_{33}}{g_{33}}\right), \quad C_{44} = \left(\frac{b_{13}d_{13}}{g_{13}} + \frac{b_{31}d_{31}}{g_{31}} + 3K_3\frac{b_{33}}{g_{33}}\right). \tag{31b–h}$$

Equation (30) can be employed to obtain numerical results for the thermal postbuckling load-deflection curves of Reissner-Mindlin plates subjected to uniform or nonuniform tent-like temperature loading and resting on softening nonlinear elastic foundations. As expected, there are three special cases: (1) if $K_3 = 0$, Equation (30) is valid for the thermal postbuckling of a Reissner-Mindlin plate resting on Winkler elastic foundations; (2) if $K_1 = K_3 = 0$, Equation (30) reduces to the thermal postbuckling equilibrium path of a Reissner-Mindlin plate without any elastic foundation, as previously given in [10]; and (3) if the plate is thin enough, then γ approaches zero and equation (30) is brought into a form suitable for the solutions of the von Kármán plate. We can readily obtain the thermal buckling load of perfect plates numerically, by setting $\mu = 0$ (or $\overline{W}^*/t = 0$), while taking $W_m = 0$ (or $\overline{W}/t = 0$). The minimum initial thermal buckling load is determined by application of Equation (30) for various values of the buckling mode (m, n), *i.e.* for different numbers of half-waves in the X- and Y- directions, respectively. From Equation (31), it can be seen that $\lambda_T^{(0)}$ only depends on the foundation stiffness K_1, thus the thermal buckling loads for Winkler and nonlinear elastic foundations are identical. As expected, the results of the next section show that the nonlinear foundation stiffness k_3 affects the thermal postbuckling response of the plate, but does not affect its linear buckling load.

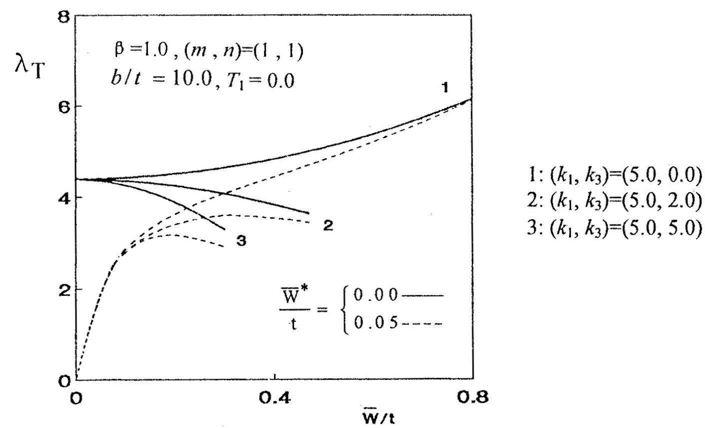

Figure 1. Thermal postbuckling load-deflection curves of Reissner-Mindlin plates on Winkler or nonlinear elastic foundations.

4. Numerical results and discussion

A thermal postbuckling analysis has been presented for the Reissner-Mindlin plates subjected to uniform or nonuniform temperature loading and resting on softening nonlinear elastic foundations. In the numerical analysis, asymptotic solutions up to 4th-order were used. A number of examples were solved to illustrate the application of the method presented. These relate to the performance of perfect and imperfect, moderately thick plates resting on softening nonlinear elastic foundations. For all of the examples (except for Table 1) $\nu = 0{\cdot}3, \alpha = 1{\cdot}0 \times 10^{-6}/°C$ and the transverse shear correction factor was considered $\kappa^2 = \pi^2/12$. Typical results are presented in dimensionless graphical form. In all figures $\overline{W}^*/t$ and $\overline{W}/t$ mean the dimensionless forms of the maximum values of the initial and additional deflection of the plate, respectively.

As part of the validation of the present method, the thermal buckling load, $\lambda_{\mathrm{T}}^* = \alpha T_0 \times 10^3$, for perfect, simply supported, isotropic square plate subjected to a uniform temperature rise with different thickness ratio and without an elastic foundation is compared in Table 1 with results of 3-dimensional solutions given by Noor and Burton [20] and of higher-order shear-deformation plate theory (HSDPT) calculated by means of the expressions given in [21] and of classical plate theory (CPT) calculated from the expressions given in [22]. Clearly, the results obtained from the present method, HSDPT and the 3-D elasticity theory are in good agreement, but CPT gives a higher buckling temperature for moderately thick and thick plates.

Figure 1 gives the thermal postbuckling load-deflection curves of Reissner-Mindlin plates subjected to a uniform temperature rise and resting on eighter Winkler or nonlinear elastic foundations. The stiffnesses for the nonlinear elastic foundation cases are $(k_1, k_3) = (5{\cdot}0, 2{\cdot}0)$ or $(5{\cdot}0, 5{\cdot}0)$ as shown and Winkler elastic foundation has $(k_1, k_3) = (5{\cdot}0, 0{\cdot}0)$. These results show that the thermal buckling loads for Winkler and nonlinear elastic foundations are identical, but that the postbuckling responses are quite different. They also show that the foundation stiffness affects the thermal postbuckling response of the Reissner-Mindlin plate significantly

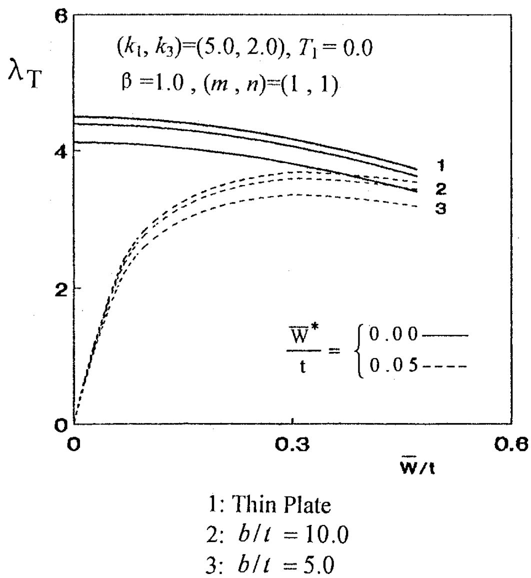

Figure 2. Effect of transverse shear deformation on the thermal postbuckling of Reissner-Mindlin plates on nonlinear elastic foundations.

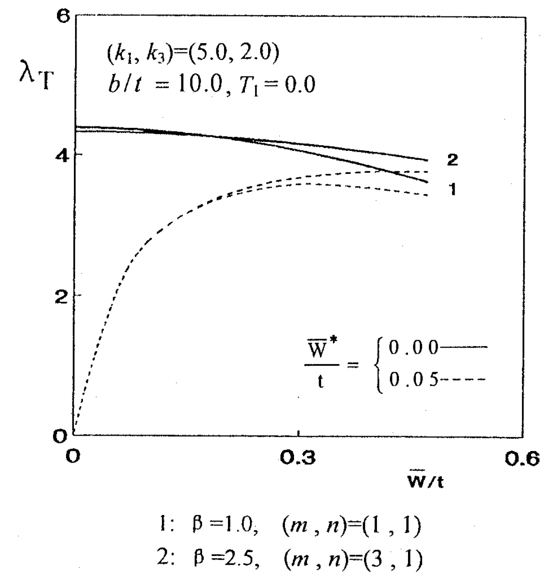

Figure 3. Effect of plate aspect ratio on the thermal postbuckling of Reissner-Mindlin plates on nonlinear elastic foundations.

and that the postbuckling equilibrium path changes from stable to unstable as the nonlinear elastic foundation stiffness k_3 increases, enabling imperfection sensitivity to be predicted.

Figure 2 gives the thermal postbuckling load-deflection curves for Reissner-Mindlin plates with different thickness ratio $b/t (= 10{\cdot}0, 5{\cdot}0)$ under uniform temperature loading and resting on a nonlinear elastic foundation and are compared with their classical counterparts. The results calculated show that the thermal buckling load of moderately thick plates with $b/t = 10{\cdot}0$ is about 2.4% lower than that of the thin plate. It is found that the thermal buckling load and postbuckling strength decrease with b/t, but they have a rather minor effect.

Figure 3 shows the effect of plate aspect ratio $\beta (= 1{\cdot}0, 2{\cdot}5)$ on the thermal postbuckling response of Reissner-Mindlin plates subjected to a uniform temperature rise and resting on a nonlinear elastic foundation. The results show that the thermal buckling loads are increased by decreasing the plate aspect ratio β, but the postbuckling deflection of plate with $\beta = 2{\cdot}5$ is larger than that of plate with $\beta = 1{\cdot}0$ when $\overline{W}/t \geqslant 0{\cdot}25$. Note that heated rectangular plate with $\beta = 2{\cdot}5$ has buckling mode with $(m, n) = (3, 1)$, whereas the square plate buckles with $(m, n) = (1, 1)$.

Figure 4 shows the effect of thermal load ratio $T_0/T_1 (= 0{\cdot}0, 0{\cdot}5, 1{\cdot}0)$ on the postbuckling response of Reissner-Mindlin plates subjected to tent-like temperature loading and resting on nonlinear elastic foundations. It can be found that the thermal buckling load decreases by increasing the thermal load ratio T_0/T_1 and that the postbuckling equilibrium path becomes significantly lower as the thermal load ratio T_0/T_1 increases.

In Figures 1, 3 and 4 the plate width-to-thickness ratio $b/t = 10{\cdot}0$ and in Figures 2–4 the foundation stiffness is characterized by $(k_1, k_3) = (5{\cdot}0, 2{\cdot}0)$.

Figure 5 shows the curves of imperfection sensitivity for heated thin or moderately thick plates resting on nonlinear elastic foundations. λ^* is the collapse load of λ_T which we made

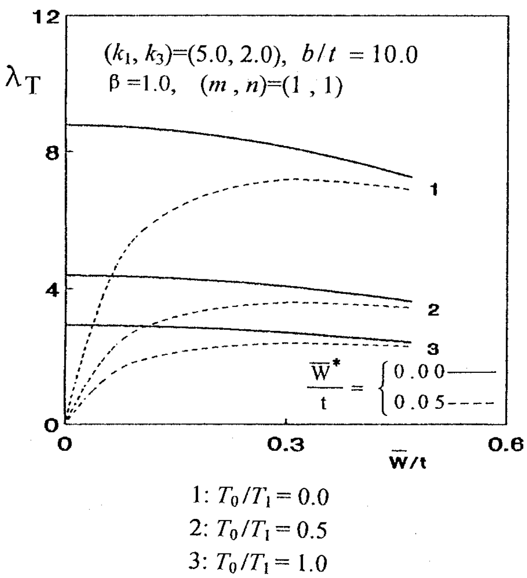

Figure 4. Effect of thermal load ratio T_0/T_1 on the postbuckling of Reissner-Mindlin plates on nonlinear elastic foundations.

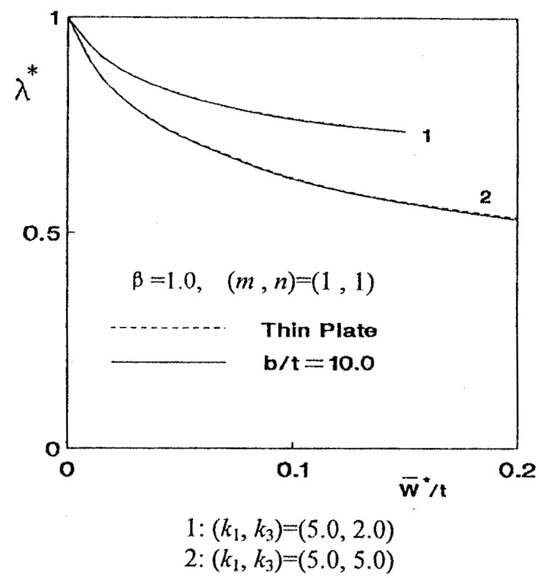

Figure 5. Comparisons of curves of imperfection sensitivity of heated thin and moderately thick plates on nonlinear elastic foundations.

dimensionless by dividing it by the critical value λ_T of the perfect plate corresponding to $\overline{W}/t = \overline{W}^*/t = 0{\cdot}0$. These results show that the imperfection sensitivity of the plate on a nonlinear elastic foundation with $(k_1, k_3) = (5{\cdot}0, 5{\cdot}0)$ is larger and is considerably greater than that of the plate on a nonlinear elastic foundation with $(k_1, k_3) = (5{\cdot}0, 2{\cdot}0)$. In contrast, the imperfection sensitivity of a thin plate is slight weaker than that of a moderately thick plate. Note that, because the reaction force p could be negative in the large-deflection range, the results presented were only for small initial geometrical imperfections.

5. Conclusions

Thermal postbuckling of simply supported, Reissner-Mindlin plates resting on softening nonlinear elastic foundations, induced by a uniform and nonuniform tent-like temperature distribution, has been studied by a perturbation method. The numerical results show that the characteristics of thermal postbuckling are influenced significantly by foundation stiffness, thermal load ratio and initial geometrical imperfection.

Unlike the plate resting on a Winkler or Pasternak-type elastic foundation, which has a stable postbuckling equilibrium path, in a number of cases a heated Reissner-Mindlin plate resting on a softening nonlinear elastic foundation, like its compressed counterpart, has an unstable postbuckling equilibrium configuration. For such cases, the plate is an imperfection-sensitive structure that exhibits all the interesting features of such structures.

Nomenclature

a, b	plate length and breadth
D	flexural rigidity for a plate
E	elastic modulus for a plate
$\overline{F}, F$	stress function and its dimensionless form
G	in-plane shear modulus for a plate
$\overline{K}_1, K_1, k_1$	Winkler elastic foundation stiffness and its two alternative dimensionless forms
$\overline{K}_3, K_3, k_3$	softening nonlinear elastic foundation stiffness and its two alternative dimensionless forms
t	thickness of a plate
$\overline{W}, W$	deflection of a plate and its dimensionless form
$\overline{W}^*, W^*$	geometrical imperfection of a plate and its dimensionless form
α	thermal expansion coefficient for a plate
β	aspect ratio of plate, $= a/b$
Δ_x, δ_x	average end-shortening and its dimensionless form
ε	a small perturbation parameter
κ^2	shear factor for a Reissner-Mindlin plate
λ^*	imperfection sensitivity parameter, $=$ (maximum λ_T of imperfect plate)/(critical λ_T of perfect plate)
λ_T	dimensionless form of thermal stress
μ	imperfection parameter
ν	Poisson's ratio

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